Duality of κ-normed topological vector spaces and their applications. *

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1 November 2000.

Abstract

In this article a duality of κ-normed topological vector spaces X is defined and investigated, where X is over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ or non-Archimedean. For such spaces the analog of the Mackey-Arens theorem is proved. A conditional κ-normability of spaces $L(X)$ of linear topological homeomorphisms of a locally convex κ-normed space X is studied, when an image of elements under the corresponding operations is in $L(X)$. There are investigated cases, when κ-normability of a topological vector space implies its local convexity. There are given applications of κ-normed spaces for resolutions of differential equations and for approximations of functions in mathematical economy.

1 Introduction.

New class of κ-metric topological spaces X was introduced earlier in works [11, 12]. κ-metric spaces may be non-metrizable, where a regular κ-metric is defined as non-negative function $\rho : X \times 2^X_o \to \mathbf{R}$ and satisfying axioms $(N1-5)$, where 2^X_o is the family of all canonical closed subsets of X and 2^X_δ is the family of all closed G_δ-subsets of X. One of the most important examples of κ-metric spaces are locally compact groups and generalized loop groups [6]. Different distance functions for subsets of linear normed spaces were studied

*Mathematics subject classification (2000): 46A03, 46A16 and 46A20

in article [1]. Topological vector spaces with κ-metrics satisfying additional conditions related with linearity of these spaces were defined and studied in work [7]. For such κ-normed spaces (see Defintion 2.1) were proved analogs of theorems about fixed point, closed graph and open mapping. Free κ-normed spaces generated by κ-metric uniform spaces with uniformly continuous κ-metrics also were studied. There were investigated categorial properties of κ-normed spaces relative to products, projective and inductive limits.

On the other hand, dual pairs and the theorems about dual topologies play very important role in the theory of topological vector spaces [8, 10]. In this work the duality of κ-normed spaces is investigated. Certainly the given definiton of duality for κ normed spaces differs from that of ordinary topological vector spaces (see Defintion 2.2). Theorem 2.4 is the development of the Mackey-Arens theorem on the case of κ-normed spaces. Theorem 2.6 is devoted to the conditional κ-normability of the uniform space $L(X)$ of linear topological homeomorphisms of the κ-normed space X. Since $L(X)$ is not the linear space, hence κ-normability of $L(X) \times 2_\delta^{L(X)}$ is considered under conditions, when the corresponding operations from Definition 2.1 do not lead outside the space $L(X) \times 2_\delta^{L(X)}$. In particular $L(X) \times 2_\delta^{L(X)}$ is the regular κ-metrizable space. Topology of κ-normed spaces is investigated in §4. There are considered cases, when the κ-normability of $(X, 2_o^X)$ leads to the κ-normability of $(X, 2_\delta^X)$. Then it is proved that the κ-normability of $(X, 2_\delta^X)$ implies local convexity of X. In §5 of this article applications of κ-normed spaces for resolutions of differential equations in infinite-dimensional over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ of complete non-normed locally convex spaces X are given. Evidently the κ-normed spaces can be used for resolutions of more complicated differential equations including partial differential equations. In §6 applications of κ-normed spaces for approximations of functions useful in mathematical economy are studied.

2 Duality of κ-normed spaces.

2.1. Defintion. A topological vector space X over the filed $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ or non-Archimedean supplied with a family either $S_X := 2_o^X$ of all canonical closed subsets or $S_X = 2_\delta^X$ of all closed G_δ-subsets is called κ-normed, if on the following product $X \times S_X$ there exists a non-negative function $\rho(x, C)$, called a κ-norm satisfying the following conditions:

$(N1)$ $\rho(x, C) = 0$ if and only if $x \in C$;

$(N2)$ if $C \subset C'$, then $\rho(x, C) \geq \rho(x, C')$;

$(N3)$ $\rho(x, C)$ is uniformly continuous by $x \in X$ for each fixed $C \in S_X$;

$(N4)$ for each increasing transfinite sequence $\{C_\alpha\}$ with $C := cl(\bigcup_\alpha C_\alpha) \in S_X$ the following equality $\rho(x, C) = \inf_\alpha \rho(x, C_\alpha)$ is satisfied, where $cl_X(A) = cl(A)$ denotes the closure of a subset A in X;

$(N5)$ (a) $\rho(x + y, cl(C_1 + C_2)) \leq \rho(x, C_1) + \rho(y, C_2)$ and

$(N5)(b)$ $\rho(x, C_1) \leq \rho(x, C_2) + \bar{\rho}(C_2, C_1)$ (or with the maximum instead of the sum on the right sides of inequalities in the non-Archimedean case), where $\bar{\rho}(C_2, C_1) := \sup_{x \in C_2} \rho(x, C_1)$;

$(N6)$ $\rho(\lambda x, \lambda C) = |\lambda| \rho(x, C)$ for each $\mathbf{K} \ni \lambda \neq 0$;

$(N7)$ $\rho(x + y, y + C) = \rho(x, C)$. If to consider the empty set $\emptyset$ as the lement of S_X, then

$(N8)$ $\rho(x, \emptyset) = \infty$ and $\rho(x, C) < \infty$ for each $x \in X$ and $\emptyset \neq C \in S_X$. As usually a subset A in X is called G_δ if it is a countable intersection of open subsets.

The space $(X, S_X) := X \times S_X$ is called κ-normed, if there is given the fixed κ-norm ρ. We denote the κ-normed space by (X, S_X, ρ).

Let X be a completely regular topological space, then the function ρ on $X \times 2^X_o$, satisfying conditions $(N1 - N4, N5(b))$ with continuity instead of uniform continuity in Axiom $(N3)$ is called the regular κ-metric. In works [11, 12] these axioms were denumerated by $(K1 - K5)$ correspondingly.

From this definition it follows, that each normed space is also κ-normed, if to put $\rho(x, C) = \inf_{y \in C} \|x - y\|$. The condition $C := cl(\bigcup_\alpha C_\alpha) \in S_X$ in Axiom $(N4)$ for $S_X = 2^X{}_o$ is satisfied automatically, since $C = cl(\bigcup_\alpha Int(C_\alpha))$. If $\{0\} \in 2^X_\delta = S_X$, then taking $\lambda \to 0$ in $(N6)$ implies the trivial equality $\rho(0, \{0\}) = 0$.

2.2. Notes and definitions. Let X and Y be two locally convex spaces over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ or the non-Archimedean spherically complete field $\mathbf{K}$ with the non-trivial multiplicative norm in this field, such that these spaces form a dual pair (X, Y). This means that a bilinear functional $< x, y >$ is given such that X separates points in Y (that is, from $< x, y >= 0$ for each $y \in Y$ it follows $x = 0$) and Y separates points in X, where $x \in X$, $y \in Y$. We suppose that the Hausdorff topology T in X is the topology of the dual pair. The latter means that Y is the continuous dual space of (X, T) (see Definition 9.6 and §9.202 [8]).

Then we define the κ-duality, that is, a mapping $<(x,A)|(y,B)>$ from $(X,S_X)\times(Y,S_Y)$ into $[0,\infty)$, satisfying the following Conditions $(D1-8)$:

$(D1)$ (a) the restriction of $<*|*>$ on $(X\times M)\times(Y\times M^\perp)$ is such that $<(x,M)|(y,M^\perp)>=|<x+M,y+M^\perp>|$, where $M\in S_X$ and $M^\perp\in S_Y$ are $\mathbf{K}$-linear subspaces, $M^\perp:=\{z\in Y: z(M)=\{0\}\}$, $<x+M,y+N>$ is the bilinear functional on $(X/M)\times(Y/N)$ for each pair of linear subspaces $M\in S_X$ and $N\in S_Y$;

$(D1)(b)$ for each $b\in Y/N$ (or $a\in X/M$), if $<a,b>=0$ for each $a\in X/M$ (or $b\in Y/N$), then $b=0$ (or $a=0$ respectively);

$(D1)(c)$ for each $x\notin A$, $A\in S_X$ (or $y\notin B$, $B\in S_Y$) there are $y\in Y$ and $B\in S_Y$ (or $x\in X$ and $A\in S_X$ correspondingly) such that $<(x,A)|(y,B)>>0$;

$(D2)$ $<(x,A_1)|(y,B_1)>\le<(x,A_2)|(y,B_2)>$ for each $A_2\subset A_1$ and $B_2\subset B_1$;

$(D3)$ $<(x,A)|(y,B)>$ is the uniformly continuous function by $x\in X$ (or $y\in Y$) for each fixed $A\in S_X$, $y\in Y$ and $B\in S_Y$ (or $x\in X$, $A\in S_X$ and $B\in S_Y$ correspondingly);

$(D4)$ $\inf_\alpha<(x,A_\alpha)|(y,B)>=<(x,A)|(y,B)>=\inf_\beta<(x,A)|(y,B_\beta)>$ for each $x\in X$, $y\in Y$ and increasing transfinite sequences $\{A_\alpha\}\subset S_X$ and $\{B_\beta\}\subset S_Y$ with $A=cl(\bigcup_\alpha A_\alpha)$, $B=cl(\bigcup_\beta B_\beta)$, $A\in S_X$, $B\in S_Y$;

$(D5)$ (a) $<(x_1+x_2,cl(A_1+A_2))|(y,B)>\le<(x_1,A_1)|(y,B)>+<(x_2,A_2)|(y,B)>$ and $<(x,A)|(y_1+y_2,cl(B_1+B_2))>\le<(x,A)|(y_1,B_1)>+<(x,A)|(y_2,B_2)>$;

$(D5)(b)$ $<(x,A)|(y,B)>\le<(x,C)|(y,B)>+\sup_{z\in C}<(z,A)|(y,B)>$ and $<(x,A)|(y,B)>\le<(x,A)|(y,E)>+\sup_{z\in E}<(x,A)|(z,B)>$ for each $x,x_1,x_2\in X$, $y,y_1,y_2\in Y$, $A,A_1,A_2,C\in S_X$, $B,B_1,B_2,E\in S_Y$ or with the maximum instead of the sum in the right sides of inequalities in the non-Archimedean case;

$(D6)$ $<(\lambda x,\lambda A)|(y,B)>=|\lambda|<(x,A)|(y,B)>=<(x,A)|(\lambda y,\lambda B)>$ for each $x\in X$, $A\in S_X$, $y\in Y$, $B\in S_Y$, $0\ne\lambda\in\mathbf{K}$;

$(D7)$ $<(x,A)|(y,B)>=<(x+a,A+a)|(y+b,B+b)>$ for each a and $x\in X$, b and $y\in Y$, $A\in S_X$, $B\in S_Y$;

$(D8)$ $<(x,A)|(y,\emptyset)>=\infty$ and $<(x,\emptyset)|(y,B)>=\infty$ for each $x\in X\setminus A$, $A\in S_X$, $y\in Y\setminus B$, $B\in S_Y$ such that $<(x,A)|(y,B)><\infty$, if $A\ne\emptyset$ and $B\ne\emptyset$.

The mapping $<(x,A)|(y,B)>$ we call the κ-form. Usually we consider $S_X=2^X_\delta$ and $S_y=2^Y_\delta$. This κ-form can be defined for 2^X_o and 2^Y_o instead of

2^X_δ and 2^Y_δ, but $(D1)(a)$ may be not satisfied, since there are locally convex spaces X and Y when all proper subspaces of X and Y may not belong to 2^X_o or 2^Y_o.

2.3. Lemma. *For each κ-duality of $(X, 2^X_\delta)$ with $(Y, 2^Y_\delta)$ from the κ-normability of $(X, 2^X_\delta)$ the κ-normability of $(Y, 2^Y_\delta)$ follows.*

Proof. Let ρ_X be a κ-norm in $(X, 2^X_\delta)$ and $M \in 2^X_\delta$ be a $\mathbf{K}$-linear subspace in X, then in accordance with Lemma 2 from [7] there is the equality $M = \bigcap_j C_j$, where $C_j \in 2^X_o$, $C_{j+1} \subset Int(C_j)$ for each j. From Axioms $(N1, 3, 5-7)$ for ρ_X it follows, that $\|x + M\|_{X/M} := \rho_X(x, M)$ is the norm in the quotient space X/M. Terefore M is equal to $M = \bigcap_n \lambda_n V$, where U is a convex balanced neighbourhood of zero in X, $U = \theta^{-1}_M\{a \in X/M : \|a\|_{X/M} < 1\}$, $\theta_M : X \to X/M$ is the quotient mapping, $V = M + U$, $0 \ne \lambda_n \in \mathbf{K}$, $\lim_n \lambda_n = 0$.

Locally convex spaces X and Y form the dual pair, hence $M^o = M^\perp = \bigcap_n \lambda_n Q^o = (\bigcup_n \lambda_n^{-1} Q)^o$, where Q is a convex balanced absorbing bounded subset in M, $Q \ni 0$, $Q^o := \{b \in Y : | < q, b > | \le 1$ for each $q \in Q\}$ is the (absolute) polar of a subset Q. Consequently, $M^\perp \in 2^Y_\delta$, since $M^\perp \subset Q^o + M^\perp \subset Q^o$, also $(Q^o + M^\perp)$ contains the unit ball $B(Y/M^\perp, 0, 1)$ from $Y/M^\perp$ and M^o is closed in Y, where

$\|b\|_{Y/M^\perp} := \sup_{0 \ne a \in X/M}(| < a, b > |/\|a\|_{X/M})$,

$B(Y, y, r) := \{z \in Y : \|z - y\|_Y \le r\}$ (see §9.3, §9.8 and §9.202 [8]). Then we put $\rho_Y(y, B) := \sup_{\rho_X(x,A) > 0} < (x, A)|(y, B) > /\rho_X(x, A)$ for each $y \in Y$ and $B \in 2^Y_\delta$. Now we need to verify that ρ_Y satisfies Axioms $(N1 - N8)$.

$(N1)$ In view of $(D1c)$ for each $y \notin B \in 2^Y_\delta$ there are $x \notin A \in 2^X_\delta$ with $< (x, A)|(y, B) >> 0$, consequently, $\rho_Y(y, B) > 0$. If $\rho_Y(y, B) = 0$, then $< (x, A)|(y, B) >= 0$ for each $(x, A) \in X \times S_X$, hence $y \in B$.

$(N2)$ In view of $(D2)$ we get: $\rho_Y(y, B_1) = \sup_{x \notin A} < (x, A)|(y, B_1) > /\rho_X(x, A) \le \sup_{x \notin A} < (x, A)|(y, B_2) > /\rho_X(x, A) = \rho_Y(y, B_2)$ for each $B_2 \subset B_1 \in 2^Y_\delta$.

The satisfaction of $(N4 - 8)$ follows from $(D4 - 8)$.

$(N3)$. Due to $(D3)$ for each $\epsilon > 0$, $x \in X \setminus A$, $A \in 2^X_\delta$ and $B \in 2^Y_\delta$ there exists a neighbourhood F of zero in Y such that $| < (x, A)|(y_1, B) > - < (x, A)|(y_2, B) > |/\rho_X(x, A) < \epsilon$ for each $y_1 - y_2 \in F$. Let $N = \bigcap_n \lambda_n Int(B)$ for $B \in 2^X_o$, where $Int(E)$ is the interior of a subset E in X. Then $N \in 2^X_\delta$ and N is the closed $\mathbf{K}$-linear subspace. Let E be a $\sigma(Y, X)$-compact disk (or a $\sigma(Y, X)$-bounded c-compact $\mathbf{K}$-disk in the non-Archimedean case) in

Y, then $E \cap N$ is a $\sigma(N,X)$-compact disk (or a $\sigma(N,X)$-bounded c-compact $\mathbf{K}$-disk correspondingly) in N. The Mackey-Arens theorem states, that for the dual pair (X,Y) the locally convex Hausdorff topology T in X is the topology of the dual pair if and only if T is the polar topology defined with the help of the family G of $\sigma(Y,X)$-compact disks (or $\sigma(Y,X)$-bounded c-compact $\mathbf{K}$-disks) in Y, which cover Y (see the Mackey-Arens theorem in §(9.6.2) and §9.202 in [8]). Then $(E \cap N)^o \supset E^o$ is the neighbourhood of zero in X, $Int(E^o) \ni 0$. We take E from the family G such that $\bigcup_{E \in \mathsf{G}} E = Y$. The Alaoglu-Bourbaki theorem states that the polar U^o is $\sigma(X',X)$-compact, if U is a neighbourhood of zero in the topological vector space X (see §§(9.3.3) and §9.202 in [8]). Then there exists $E \in \mathsf{G}$ for which $N^{\perp} = N^o = \bigcap_n \lambda_n (E \cap N)^o = (\bigcup_n \lambda_n^{-1}(E \cap N))^o$ and $Int(E^o) \ni 0$ due to the Alaoglu-Bourbaki theorem and the equality $E^{oo} = cl_{\sigma(Y,X)} E$ in accordance with §(9.3.2) and §9.202 [8]. Consequently, $N^{\perp} \in 2^X_{\delta}$. Then $\rho_X(x, N^{\perp})$ is the norm in $X/N^{\perp}$ and $\rho_Y(y,N)$ is the norm in Y/N. Since $B + N = B$, then due to $(D1)(a,b)$ we have that $\rho_Y(y,B)$ is uniformly continuous by $y \in Y$, since due to $(D3)$ and $(N2,6,7)$ there exists a constant $C > 0$ such that $| < (x,A)|(y_1,B) > - < (x,A)|(y_2,B) > |/\rho_X(x,A) \le C\rho_Y(y_1 - y_2, N)$, if a neighbourhood F is given with the help of the norm $\rho_Y(y,N)$ in Y/N and the quotient mapping $\theta_N : Y \to Y/N$ (see §6.2.2 and §6.205 in [8]). Lemma 2 [7] states that for the κ-mertizable space $(X, 2^X_{\delta})$ for each $C \in 2^X_{\delta}$ there exists a sequence $C_j \in 2^X_o$ such that $Int(C_j) \supset C_{j+1}$ for each $j \in \mathbf{N}$ and $\bigcap_j C_j = \bigcap_j Int(C_j) = C$. In accordance with the Corollary 5 [7] for the κ-normed locally convex space X for each $C \in 2^X_{\delta}$ and $\epsilon > 0$ there exists a convex balanced neighbourhood $U \ni 0$ such that

$0 \le \rho(x,C) - \rho(x, cl(C+U)) < \epsilon$ for each $x \in X$. Vice versa the latter property for the continuous κ-norm without Axiom $(N3)$ implies its uniform continuity. Then if $x \in B \in 2^X_{\delta}$, then $0 \in (B - x) \in 2^X_{\delta}$. With the help of Axiom $(N4)$, Lemma 2 and Corollary 5 from [7] and the equality $(\bigcup_{n=1}^{\infty} C_n)^o = \bigcap_{n=1}^{\infty} (C_n^o)$ while $C_n \in 2^X_0$ for each n we get that $(N3)$ is satisfied on $S_Y = 2^Y_{\delta}$. If $0 \in E \in 2^Y_{\delta}$, then for the locally convex space Y from $E = \bigcap_{n=1}^{\infty} E_n$ with $0 \in E_n \subset cl(E_n) \in 2^Y_o$ and open E_n it follows, that there are convex balanced open subsets V_n in E_n for each n. Then $\bigcap_n V_n =: V$ is the convex balanced G_{δ}-subset in Y, moreover, $V \subset E$, consequently, the linear subspace $\bigcap_n \lambda_n V =: M$ is the closed G_{δ}-subset in Y, since $M \subset E$, where $\lim_{n\to\infty} \lambda_n = 0$, $0 \ne \lambda_n \in \mathbf{K}$ for each $n \in \mathbf{N}$.

Then we get the following stronger result, where the supposition about

existence of κ-duality is omitted.

2.4. Theorem. *Let X and Y be a dual pair of locally convex spaces and let $(X, 2^X_\delta)$ be κ-normable, then $(Y, 2^Y_\delta)$ is also κ-normable.*

Proof. In view of Lemma 2.3 it is sufficient to construct a κ-duality. In view of the Mackey-Arens theorem the base B of topology T on X is given by polars E^o, $E \in \mathsf{G}$, where G is a family of $\sigma(Y,X)$-compact diks (or $\sigma(Y,X)$-bounded c-compact $\mathbf{K}$-disks) in Y, for which $\bigcup_{E\in\mathsf{G}} E = Y$. In view of the Weak Representation Theorem (9.2.3) and §9.202 [8] for each continuous linear functional g on X there is the unique element $y \in Y$ such that $g(x) =< x, y >$. Let

(i) $< (x,A)|(y,B) >:= \inf_{(a\in A, b\in B)} | < (b-y),(a-x) > |$ for each $x \in X$, $\emptyset \ne A \in 2^X_\delta$, $y \in Y$, $\emptyset \ne B \in 2^Y_\delta$ and $< (x,A)|(y,B) >= \infty$ for $A = \emptyset$ or $B = \emptyset$ with $x \notin A$ and $y \notin B$. For the proof it is sufficient to verify that Conditions $(D1 - D8)$ are satisfied.

$(D1)(a)$. If M and $N = M^\perp$ are $\mathbf{K}$-linear subspaces in X and Y, $M \in 2^X_\delta$, $N \in 2^Y_\delta$, then $< (x,M)|(y,N) >= \inf_{(a\in M, b\in N)} | < (b-y),(a-x) > | =< (x+c,M)|(y+d,N) >$ for each $c \in M$ and $d \in N$, $< (x,M)|(y,N) >= | < \theta_N(y), \theta_M(x) > |$, where $\theta_M : X \to X/M$ and $\theta_N : Y \to Y/N$ are quotient mappings such that $< x+M, y+N >=< \theta_N(y), \theta_M(x) >$.

$(D1)(b)$. Since Y separates points in X, and X in Y, then the same is true for the pair $(X/M, Y/N)$, since $N^\perp = M^{\perp\perp} = M$.

$(D1)(c)$. If $E \in 2^X_\delta$, then $z + E \in 2^X_\delta$ for each $z \in X$. If $x \in X \setminus A$, $A \in 2^X_\delta$ is convex and balanced (each $E \in 2^X_\delta$ with $0 \in E$ contains also such subset A due to local convexity of X, if $E \in 2^X_0$, we get $A \in 2^X_o$), then $M = \bigcap_n \lambda_n A$ is the closed $\mathbf{K}$-linear subspace, $x \notin M$, $\theta_M(A)$ is bounded in X/M. There exists a ball S of radius $\epsilon > 0$ with the centre in $x + M$ in the normed space X/M such that $S \cap \theta_M(A) = \emptyset$. In view of the Hahn-Banach theorem (see §(8.4.6) and §8.203 in [8]) for the locally convex space X over $\mathbf{R}$, $\mathbf{C}$ or the spherically complete non-Archimedean field $\mathbf{K}$ and the linear subspace Z in X each continuous linear functional on Z has a continuous extension on X. Therefore, there exists a continuous linear functional $y \in Y$ such that $y(x) \notin y(A)$, then there exists a neighbourhood $Int(B) \ni y$ such that $B \in 2^Y_o$ for which $B(x) \cap B(A) = \emptyset$, since the field $\mathbf{K}$ is one of the following: $\mathbf{R}$, $\mathbf{C}$ or the spherically complete non-Archimedean field, where $B(A) := \{b(a) : b \in B, a \in A\}$ for $A \subset X$ and $B \subset Y$. Since cokernels of linear functionals are one-dimensional over $\mathbf{K}$, then there are $B \in 2^Y_\delta$ with $< (x,A)|(0,B) >> 0$. If $x \in A$ or $y \in B$, then $< (x,A)|(y,B) >= 0$, since

$0(A-x)=\{0\}$, $(B-y)(0)=\{0\}$.

$(D2)$. $\inf_{(a\in A_1, b\in B_1)} |<(b-y),(a-x)>| \le \inf_{(a\in A_2,b\in B_2)} |<(b-y),(a-x)>|$ for each $A_2\subset A_1\in 2^X_\delta$ and $B_2\subset B_1\in 2^Y_\delta$.

$(D4)$. For each $a\in A$ and $b\in B$ there exist transfinite sequences of elements $a_\alpha\in A_\alpha$ and $b_\beta\in B_\beta$ such that a_α converges to a, b_β to b. From the separate continuity of $<(b-y),(a-x)>$ by $a\in X$ and also by $b\in Y$ it follows $(D4)$. Conditions $(D5-8)$ follow from definitions.

$(D3)$. In view of $(D1)$ it is sufficient to consider the case of the pair $(X/M, Y/M^\perp)$ of normed spaces and bounded subsets $A\in 2^{X/M}_\delta$ and $B\in 2^{Y/M^\perp}_\delta$, that is, $A\subset B(X/M,0,r_1)$, $B\subset B(Y/M^\perp,0,r_2)$, $0<r_1<\infty$, $0<r_2<\infty$. The topological vector space X over the non-Archimedean field $\mathbf{K}$ with the non-trivial norm or a convex balanced subset E in X is called c-closed, if each $\mathbf{K}$-convex balanced base of the filter in X or in E respectively has a limit point. In view of the Alaoglu-Bourbaki theorem $(A-x)$ and $(B-y)$ are $\sigma(X,Y)$ and $\sigma(Y,X)$-compact (or $\sigma(X,Y)$ and $\sigma(Y,X)$-bounded and c-compact in the non-Archimedean case) respectively (see §(5.6.2), §5.204 [8] and Theorem 3.1.2 [13]). Since the linear functionals $(a-x)\in(A-x)$ and $(b-y)\in(B-y)$ are continuous in the weak topologies $\sigma(Y,X)$ and $\sigma(X,Y)$ on Y and X correspondingly, then the value $\inf_{(a\in A,b\in B)}|<(b-y),(a-x)>|$ of $<(b-y),(a-x)>$ is attained on the corresponding elements $a_0\in A$ and $b_0\in B$ due to Theorem 3.1.10 [13] and in the non-Archimedean case due to §5.204 [8]. From the inequality $|z(g)|\le \|z\|_{Y/M^\perp}\|g\|_{X/M}$ it follows, that $|\inf_{(a\in A,b\in B)}|<(b-y_1),(a-x)>|-\inf_{(a\in A,b\in B)}|<(b-y_2),(a-x)>||\le \|y_1-y_2\|_{Y/M^\perp}r_1$ and $|\inf_{(a\in A,b\in B)}|<(b-y),(a-x_1)>|-\inf_{(a\in A,b\in B)}|<(b-y),(a-x_2)>||\le \|x_1-x_2\|_{X/M}r_2$. Moreover, $\rho_X(x+M,M)$ is the continuous norm in X/M, that is, there exists a constant $C>0$, for which $\rho_X(x,M)\le C\|x+M\|_{X/M}$ for each $x\in X$, where $\theta_M(x)=x+M\in X/M$. Since $\inf_{(a\in A,b\in B)}|<(b-y),(a-x)>|$ is invariant under the substitution of x on $x+f$ with $f\in M$, y and $y+q$ with $q\in M^\perp$, then $<(x,A)|(y,B)>$ satisfies Conditon $(D3)$. As at the end of §3 the κ-norm ρ_Y has an extension from $(Y,2^Y_o)$ on $(Y,2^Y_\delta)$.

2.5. Remark. Vice versa, if there is given ρ_Y in $(Y,2^Y_\delta)$ and κ-duality by the formula (i), then the κ-norm

(i) $\tilde\rho_X(x,A) := \sup_{\rho_Y(y,B)>0} <(x,A)|(y,B)>/\rho_Y(y,B)$ is equivalent to ρ_X, that is, by the definition there exist constants $0<C_1<C_2<\infty$, for which $C_1\tilde\rho_X(x,A)\le\rho_X(x,A)\le C_2\tilde\rho_X(x,A)$ for each $A\in 2^X_\delta$ and $x\in X$.

Indeed, for $\mathbf{K}$-linear subspaces $A = M \in 2^X_\delta$ and $B = M^\perp \in S_Y$ this is evident from the Hahn-Banach theorem, since $\|x+M\|_{X/M} = \rho_X(x,M)$. On the other hand, from Formulas 2.4.(i) and 2.5.(i) it follows, that $\tilde{\rho}_X(x,A) \le \rho_X(x,A)$ for each $A \in 2^X_\delta$ and $x \in X$. Let $M = \bigcap_n \lambda_n A$ for a convex balanced subset A, where $0 \ne \lambda_n \in \mathbf{K}$ for each $n \in \mathbf{N}$ and $\lim_{n\to\infty} \lambda_n = 0$. Then $\theta_M(A)$ is bounded in X/M, consequently, for each $\epsilon > 0$ there exists $a \in A$ such that $|\|a-x+M\|_{X/M} - \rho_X(x,A)| < \epsilon$. In view of the Hahn-Banach theorem there exists $y \in Y$ such that $\theta_{M^\perp}(y) \in Y/M^\perp$ and $|y(a-x)| = \|a-x+M\|_{X/M}$, moreover, $|y(x+M)| \le \|x+M\|_{X/M}$ for each $x \in X$, where $\theta_M : X \to X/M$ and $\theta_{M^\perp} : Y \to Y/M^\perp$ are the quotient mappings. Therefore, ρ_X and $\tilde{\rho}_X$ are equivalent κ-norms.

Certainly, as in the case of normed spaces this does not mean that X is necessarily reflexive.

2.6. Let X be a locally convex space and $L(X)$ be an uniform space of linear topological homeomorphisms $S : X \to X$ supplied with the base $W(U,V)$ of topology induced from the space $\mathsf{O}(X)$ of linear continuous operators from X into X such that $W(U,V) := \{S \in \mathsf{O}(X) : S(U) \subset V\}$, where U and $V \in \mathsf{V}$, V is the base of neighbourhoods of zero in X.

We say that $L(X)$ is conditionally κ-normable if and only if there exists a κ-norm, for which Axioms $(N1-N8)$ are satisfied, when results of operations $(A,B) \mapsto A+B$; $(\lambda, A) \mapsto \lambda A$; $(S_1+S_2) \mapsto cl(S_1+S_2)$; $(A,S) \mapsto A+S$; $(\lambda, S) \mapsto \lambda S$ and $cl(\bigcup_\alpha S_\alpha) = S$ for increasing transfinite sequences belong to $L(X)$ and $2^{L(X)}_\delta$ correspondingly, where A and $B \in L(X)$, S and $S_\alpha \in 2^{L(X)}_\delta$, $0 \ne \lambda \in \mathbf{K}$.

Theorem. *From the κ-normability of $(X, 2^X_\delta)$ the conditional κ-normability of $(L(X), 2^{L(X)}_\delta)$ follows, when X is over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ or the non-Archimedean spherically complete field with the non-trivial norm.*

Proof. Each locally convex space X is isomorphic with the projective limit of normed spaces Y_α, that is, $X = pr-\lim\{Y_\alpha, \pi^\alpha_\beta, \Lambda\}$, where $\pi^\alpha_\beta : Y_\alpha \to Y_\beta$ are continuous linear mappings from Y_α onto Y_β for each $\alpha > \beta \in \Lambda$, Λ is a directed set, $\pi^\alpha_\beta \circ \pi^\beta_\gamma = \pi^\alpha_\gamma$ for each $\alpha \ge \beta \ge \gamma \in \Lambda$, $\pi_\alpha : X \to Y_\alpha$ are quotient mappings (see §2.5 [13], §6.5, Theorem (6.7.2), §6.205 [8]). Let p_α be a seminorm in X such that $X_\alpha = p^{-1}(0)$ and $Y_\alpha = X/X_\alpha$. The seminorm p_α induces the norm $\hat{p}_\alpha$ in Y_α, then $X_\alpha = \bigcap_n \lambda_n Int(U_\alpha)$, where $U_\alpha = \pi^{-1}_\alpha(B(X_\alpha,0,1))$, $0 \ne \lambda_n \in \mathbf{K}$ for each $n \in \mathbf{N}$ and $\lim_n \lambda_n = 0$, $B(Y_\alpha,0,1)$ is the ball of radius 1 with center 0 in Y_α. Therefore, $X_\alpha \in 2^X_\delta$

for each $\alpha \in \Lambda$. If M is a closed $\mathbf{K}$-linear subspace in X, then $A(M)$ is linearly topologically isomorphic with M for each $A \in L(X)$. The operator A generates the opearator $\dot{A} : X/M \to X/M$ such that $\dot{A} \in L(X/M)$ and $\dot{A}(x+M) = A(x)+A(M)$. At the same time X/M is the normed space with the norm $\rho_X(x,M)$, where $x+M = \theta_M(x)$ and $\theta_M : X \to X/M$ is a quotient mapping. Then the topolgy in $L(X/M)$ is hereditary from the normed space $\mathsf{O}(X/M)$. Evidently $\mathbf{K}^* L(X) = L(X)$, where $\mathbf{K}^* := \mathbf{K} \setminus \{0\}$. Let $\rho_L(A,S) := \sup_{(x \in X \setminus E, E \in 2^X_o)} \inf_{(B \in S, a \in E)} (\|(\dot{B} - \dot{A})(x - a + M)\|_{X/M} / \rho_X(x,E))$ for each $S \in 2^{L(X)}_o$, $S \neq \emptyset$, where $M = \bigcap_n \lambda_n E$, $\dot{A}$ and $\dot{B}$ are operaators on X/M, a subset E is convex and balanced, $0 \neq \lambda_n \in \mathbf{K}$ for each $n \in \mathbf{N}$ such that $\lim_{n\to\infty} \lambda_n = 0$. If $S = \emptyset$ we put $\rho_L(A,\emptyset) = \infty$. It remains to verify Conditions $(N1{-}8)$, when the results of operations $(A,B) \mapsto A+B$; $(\lambda, A) \mapsto \lambda A$; $(S_1+S_2) \mapsto cl(S_1+S_2)$; $(A,S) \mapsto A+S$; $(\lambda,S) \mapsto \lambda S$ and $cl(\bigcup_\alpha S_\alpha) = S$ for increasing transfinite sequences belong to $L(X)$ and $2^{L(X)}_\delta$ respectively, where A and $B \in L(X)$, S and $S_\alpha \in 2^{L(X)}_\delta$, $0 \neq \lambda \in \mathbf{K}$.

$(N1)$. If $A \in S$, then from $0(x-E) = \{0\}$ it follows that $\rho_L(A,S) = 0$. If $A \notin S$, then there exists a convex balanced neigbourhood $W(U,V)$ of zero in $\mathsf{O}(X)$ such that $(A + W(U,V)) \cap S = \emptyset$, where $U = \bigcap_{j=1}^n \pi^{-1}_{\alpha_j}(B(Y_{\alpha_j},0,r_j))$, $n \in \mathbf{N}$, $0 < r_j < \infty$, $V = \bigcap_{i=1}^m \pi^{-1}_{\beta_i}(B(Y_{\beta_i},0,R_i))$, $m \in \mathbf{N}$, $0 < R_i < \infty$. In view of the Hahn-Banach theorem there are $f \in X'$ and $g \in X$ such that $q_{A,S}(f,g) := \inf_{B\in S} |f(B-A)g| > 0$, since $W(U,V)F$ is convex in X for each convex neighbourhood F of zero in X. Then in view of continuity of $q_{A,S}(f,g)$ by f and g there exists a neighbourhood U such that $\inf_{h\in U} q_{A,S}(f,h+g) > 0$, consequently, $\inf_{(B\in S, a\in E)} \|(\dot{B} - \dot{A})(a+M)\|_{X/M} > 0$, where $E = g + U$.

$(N2)$. Follows from the inequality $\inf_{(B\in S_1, a\in E_1)} \|(\dot{B}-\dot{A})(x-a+M)\|_{X/M} \le \inf_{(B\in S_2, a\in E_2)} \|(\dot{B} - \dot{A})(x - a + M)\|_{X/M}$ for each $S_2 \subset S_1$.

$(N3)$. Since X/M and $\mathsf{O}(X/M)$ are normed spaces, then $\inf_{(B\in S, a\in E)} \|(\dot{B} - \dot{A})(x - a + M)\|_{X/M}$ is uniformly continuous by $\dot{A} \in L(X/M)$ for fixed $S \in 2^{L(X)}_o$, $x \in X$ and $E \in 2^X_o$. Moreover, there exists $C = const > 0$ such that $\rho_X(x,M) \le C\|x+M\|_{X/M}$ for each $x \in X$. Consequently, for each $\epsilon > 0$, $x \in X \setminus E$, $E \in 2^X_o$, $S \in 2^{L(X)}_o$ there exists a neighbourhood of zero F in $\mathsf{O}(X)$ such that $|\inf_{(B\in S,a\in E)} \|(\dot{B}-\dot{A}_1)(x-a+M)\|_{X/M} - \inf_{(B\in S,a\in E)} \|(\dot{B} - \dot{A}_2)(x - a + M)\|_{X/M}| / \rho_X(x,E) < \epsilon$ for each $A_1 - A_2 \in F$.

$(N4)$. For each $s \in S = cl(\bigcup_\alpha S_\alpha)$ there exists a transfinite sequence $s_\alpha \in S_\alpha$, converging to $s \in S$, where S and $S_\alpha \in 2^{L(X)}_o$ for each α, $\{S_\alpha : \alpha\}$ is the increasing transfinite sequence. Then for each $x \in X$ and $E \in 2^X_o$

there exists $\lim_\alpha (\inf_{a\in E} \|(s_\alpha - \dot{A})(x-a+M)\|_{X/M}) = \inf_{a\in E} \|(s-\dot{A})(x-a+M)\|_{X/M}$, consequently, $\lim_\alpha (\inf_{b\in S_\alpha, a\in E)} \|(\dot{B}-\dot{A})(x-a+M)\|_{X/M}) = \inf_{(B\in S, a\in E)} \|(\dot{B}-\dot{A})(x-a+M)\|_{X/M}$, since $|\|\dot{B}_1(y+M)\|_{X/M} - \|\dot{B}_2(y+M)\|_{X/M}| \le \|\dot{B}_1 - \dot{B}_2\|_{\mathsf{O}(X/M)} \|y+M\|_{X/M}$.

$(N5)$. For each $x\in X$, A, B and $C \in \mathsf{O}(X)$ we have $\|(\dot{A}-\dot{B})(x+M)\|_{X/M} \le \|(\dot{A}-\dot{C})(x+M)\|_{X/M} + \|(\dot{B}-\dot{C})(x+M)\|_{X/M}$, hence $\inf_{(B\in cl(S_1+S_2), a\in E)} \|(\dot{B}-\dot{A}_1-\dot{A}_2)(x-a+M)\|_{X/M} \le \inf_{(B_1\in S_1, a\in E)} \|(\dot{B}_1-\dot{A}_1)(x-a+M)\|_{X/M} + \inf_{(B_2\in S_2, a\in E)} \|(\dot{B}_2-\dot{A}_2)(x-a+M)\|_{X/M}$ and inevitably $\inf_{(B\in S_1, a\in E)} \|(\dot{B}-\dot{A})(x-a+M)\|_{X/M} \le \inf_{(C\in S_2, a\in E)} \|(\dot{C}-\dot{A})(x-a+M)\|_{X/M} + \sup_{C\in S_2} \inf_{(B\in S_1, a\in E)} \|(\dot{B}-\dot{C})(x-a+M)\|_{X/M}$ (with the maximum instead of the sum in the right parts of the inequalities in the non-Archimedean case). Taking $\sup_{(x\in X\setminus E, E\in 2^X_o)}$ of both parts of inequalities divided by $\rho_X(x,E)$ we get $(N5)$.

$(N6-8)$ follow from the definition of ρ_L. As at the end of §3 the κ-norm has the continuous extension from $L(X)\times 2_o^{L(X)}$ on $L(X)\times 2_\delta^{L(X)}$.

3 Features of a κ-normed space topology.

3.1. Remark. Let X be a locally convex space such that $(X, 2^X_\delta)$ is κ-normable by a κ-norm ρ. The topology of X is defined by a family of seminorms p_α. Then $X_\alpha := p_\alpha^{-1}(0)$ are closed $\mathbf{K}$-linear subspaces in X. In view of the κ-normability we have as above $X_\alpha \in 2^X_\delta$ such that $\rho_\alpha(x) := \rho(x, X_\alpha)$ are seminorms in X. Each quotient space X/X_α is normable by a norm $\hat{p}_\alpha([x]) := \inf_{z\in X_\alpha} p_\alpha(x+z)$, where $x\in [x] := x+X_\alpha$. Due to uniform continuity of $\rho(x,C)$ by $x\in X$ for each fixed $C\in 2^X_\delta$ we have that the family $\{\rho_\alpha\}$ defines topology of X and ρ_α are seminorms in X. In view of $(N1)$ we get that $\hat{\rho}_\alpha([x]) := \rho_\alpha(x, X_\alpha)$ is the norm in the quotient space X/X_α for each α, where $x\in [x] := x+X_\alpha$. Therefore, we can choose a family

$$(i)\ \{p_\alpha : \alpha\in\Omega\}$$

defining topology of X such that

$$(ii)\ p_\alpha(x) \le \hat{\rho}_\alpha(x)$$

for each $\alpha\in\Omega$ and $x\in X$.

Theorem. *Let $(X, 2^X_\delta, \rho)$ be the κ-normed space, whose topology is defined by Family (i) of seminorms satisfying Condition (ii), where*

$$(iii)\ \sup_{\alpha\in\Omega}(x, X_\alpha) < \infty$$

for each $\alpha \in \Omega$. Then for each countable subfamily $\{p_\alpha : \alpha \in \Lambda\}$, where $\Lambda \subset \Omega$ and $card(\Lambda) \le \aleph_0$, the following function $q_\Lambda(x) := \sup_\alpha p_\alpha(x)$ is a continuous seminorm in X.

Proof. In view of Conditions (ii, iii) we have $q_\Lambda(x) < \infty$ for each $x \in X$. Each p_α is non-negative, hence $0 \le q_\Lambda(x)$ for each $x \in X$. Since each $p_\alpha(x)$ is a seminorm, then $q_\Lambda(x + y) = \sup_{\alpha\in\Lambda} p_\alpha(x + y) \le sup_{\alpha\in\Lambda} p_\alpha(x) + \sup_{\beta\in\Lambda} p_\beta(x) = q_\Lambda(x) + q_\Lambda(y)$ for each $x, y \in X$, which is the triangle inequality for q_Λ. In the non-Archimedeean case we have the strong triangle inequality $q_\Lambda(x + y) \le \max(q_\Lambda(x), q_\Lambda(y))$. Suppose that for some countable subset Λ the seminorm q_Λ is not continuous in X. We consider the subspace $Y := \bigcap_{\alpha\in\Lambda} X_\alpha$, then $Y \in 2^X_\delta$, since $Y = \bigcap_{j=1}^\infty(\bigcap_{\alpha\in\Lambda} \lambda_j p_\alpha^{-1}([0, 1]))$, where $\lambda_j \ne 0$ for each $j \in \mathbf{N}$ and $\lim_j \lambda_j = 0$. If q_Λ is not continuous, then $Int_X q_\Lambda^{-1}([0, 1]) = \emptyset$, since $q_\Lambda^{-1}([0, 1])$ is absolutely convex in X, consequently, there exists a vector $0 \ne v \in X\setminus Y$ such that $\rho(v, Y) > 0$. On the other hand, $\rho(x, Y)$ is uniformly continous by $x \in X$, consequently, there exists a seminorm p such that $\rho(x, Y) = p(x)$ for each $x \in X$. But in view of Conditions $(i - iii)$ and Axiom $(N8)$ we have $q_\Lambda(x) \le \sup_{\alpha\in\Lambda} \rho(x, X_\alpha) \le \rho(x, Y) < \infty$ for each $x \in X$. This inequality contradicts our supposition that q_Λ is discontinuous, consequently, q_Λ is continuous.

3.2. Theorem. *Let X be a locally convex space and let $(X, 2^X_o, \rho)$ be a κ-normed space. Suppose that a topology of X is defined by a family of seminorms $\{p_\alpha : \alpha \in \Omega\}$ such that*

(i) $q_\Lambda(x) < \infty$ for each $x \in X$ and $q_\Lambda(x)$ is continuous by $x \in X$ for each countable subset $\Lambda \subset \Omega$, where $q_\Lambda(x) := \sup_{\alpha\in\Lambda} p_\alpha(x)$. Then ρ has the extension on $X \times 2^X_\delta$.

Proof. Let $X_\alpha := p_\alpha^{-1}(0)$, then we have a κ-norm $\rho_\alpha(x, cl(C + X_\alpha)) =: \hat{\rho}_\alpha([x], [C])$ in $(Y_\alpha, 2^{Y_\alpha}_o)$ for each α, where $Y_\alpha := X/X_\alpha$ is the quotient space, $[x] = x + X_\alpha$, $[C] := cl(C + X_\alpha)$. Each Y_α is the normed space with the norm $\hat{p}_\alpha([x]) := \inf_{(z\in X_\alpha)} p_\alpha(x + z)$, where $x \in [x]$. To finish the proof we need the following lemma.

3.3. Lemma. *Let Y be a normed space and ρ be a κ-norm on $Y \times 2^Y_o$, then ρ has an extension on $Y \times 2^Y_\delta$.*

Proof. Let p be a norm in Y, then $p^{-1}([0,1/n])$ is a canonical closed neighbourhood of zero in Y such that $\{0\} = \bigcap_{n=1}^{\infty} p^{-1}([0,1/n]) \in 2^Y_\delta$. Put $\rho(x,\{0\}) := \sup_{n\in\mathbf{N}} \rho(x, p^{-1}([0,1/n]))$. For each $n \in \mathbf{N}$ according to Axiom $(N2)$ we have the following inequality: $\rho(x,p^{-1}([0,1/(n+1)])) \ge \rho(x,p^{-1}([0,1/n]))$. In view of Axiom $(N5)$ we have another inequality: $\rho(x,p^{-1}([0,1/(n+k)])) - \rho(x,p^{-1}([0,1/n])) \le \bar{\rho}(p^{-1}([0,1/n]), p^{-1}([0,1/(n+k)]))$, where $\bar{\rho}(A,B) := \sup_{a\in A}\rho(a,B)$ for each $A, B \in 2^Y_0$ and $k \in \mathbf{N}$. In view of Corollary 5 [7] for each $\epsilon > 0$ there exists $n \in \mathbf{N}$ such that $\bar{\rho}(p^{-1}([0,1/n]), p^{-1}([0,1/(n+k)])) < \epsilon$ for each $k \in \mathbf{N}$. Therefore $\rho(x,\{0\})$ is the continuous norm in Y. Each norm in Y induces a κ-norm in $(Y, 2^Y_\delta)$, consequently, ρ has an extension on $Y \times 2^Y_\delta$.

Continuation of the **proof** of Theorem 3.2. In view of Lemma 3.3 we have an extension $\hat{\rho}_\alpha$ on $Y_\alpha \times 2^{Y_\alpha}_\delta$. Due to Proposition 13 [7] there exists a locally convex space Z and a κ-norm η in $(Z, 2^Z_\delta)$ such that each Y_α is the quotient space of Z and each κ-norm η_{Y_α} in $(Y_\alpha, 2^{Y_\alpha}_\delta)$ is equivalent to $\hat{\rho}_\alpha$. From the construction of Z (see Conditions $(a-d)$ in the proof of Proposition 13 [7]) and Condition (i) of Theorem 3.2 we have that X is the quotient space of Z as the locally convex space, since each $\hat{\rho}_\alpha(x,C)$ is uniformly continuous by $x \in Y_\alpha$ for each fixed $C \in 2^{Y_\alpha}_\delta$ relative to the norm $\hat{p}_\alpha$. In view of Theorem 6 [7] the κ-norm η in Z induces a κ-norm $\tilde{\rho}$ in $(X, 2^X_\delta)$. The restrictions of these two κ-norms on $X \times 2^X_o$ are equivalent and $\hat{\tilde{\rho}}_\alpha(x,C) = \hat{\rho}_\alpha$ for each $x \in Y_\alpha$ and $C \in 2^{Y_\alpha}_o$.

3.4. Remark. In the case of X equal to the countable product of normed spaces Condition 3.2.(i) produces the box topology.

3.5. Corollary. *Let X be a complete locally convex space satisfying Condition* 3.2.(i) *with a κ-norm ρ in $(X, 2^X_o)$, which induces a metric D in 2^X_o in accordance with Definition 7 [7]. Then the completion of 2^X_o is equal to 2^X_δ.*

Proof. In view of Theorem 3.2 ρ has the extension on $X \times 2^X_\delta$. In accordance with Theorem 9 [7] $(2^X_\delta, D)$ is complete, where D has the natural extension on $(2^X_\delta)^2$. In view of Corollary 5 [7] we have that 2^X_o is dense in 2^X_δ.

3.6. Theorem. *Let $(X, 2^X_\delta, \rho)$ be a κ-normed topological vector space. Then X is locally convex.*

Proof. Let us suppose that X is not locally convex and we consider a base B_o of neighbourhoods of zero in X. Without loss of generality it can be supposed that each $C \in \mathsf{B}_o$ is balanced. Since X is not locally convex

there exists $C \in \mathsf{B}_o$ such that $\bigcap_{\lambda_j} \lambda_j C = E$ is not the $\mathbf{K}$-linear subspace of X, where $\lambda_j \ne 0$ for each $j \in \mathbf{N}$ and $\lim_j \lambda_j = 0$, where X is over the field $\mathbf{K}$. If $x \in E$, then $\lambda_j x \in E$ for each j, consequently, $\mathbf{K}x \subset E$. Indeed, if E is the $\mathbf{K}$-linear subspace, then E is closed in X, since from $cl(\alpha E) \subset \beta E$ for each $0 < |\alpha| < |\beta|$ it follows, that for E it can be chosen $\lambda_j \ne 0$ such that $cl(\lambda_{j+1}C) \subset \lambda_j C$ for each $j \in \mathbf{N}$. Therefore, X/E would be the $\mathbf{K}$-linear topological space. In view of Theorem 6 [7] the quotient space X/E is κ-normed with the κ-norm $\rho_{X/E}$. But $C + E = C$ in X and $\pi_E(C+E)$ is bounded in X/E, where $\pi_E : X \to X/E$ is the quotient mapping. Since $E \in 2^X_\delta$, then $\rho_{X/E}([x], \{0\})$ is the norm in X/E, where $x \in X$, $[x] = x + E = \pi_E(x) \in X/E$, $[E] = \{0\} \in X/E$. So it would mean that X/E is the normed space, hence X would be a projective limit of normed spaces, consequently, X is locally convex. This contardicts our assumption.

This means that there exists $C \in \mathsf{B}_o$ or $C \in 2^X_\delta$ for which E is not the linear subspace of X. Therefore, there exists a system of vectors $\{x_j : j \in \alpha\}$ in E such that $sp_{\mathbf{K}}\{x_j : j \in \beta\}$ is not contained in E for each subset β in α, where $\{x_j : j \in \alpha\}$ is a system of $\mathbf{K}$-linearly independent vectors in X. If $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$ we take $\mathbf{L} := \mathbf{K}$. We can consider the completion of X, which is also κ-normed due to uniform continuity of ρ. So we can suppose that X is complete. Then $\mathbf{K}$ has to be uniformly complete for complete X. If $\mathbf{K}$ is the non-Archimedean non-locally compact field, then it has a locally compact subfield $\mathbf{L}$ such that X can be considered as the $\mathbf{L}$-linear space $X_{\mathbf{L}}$. Spaces X and $X_{\mathbf{L}}$ are uniformly homeomorphic and $\mathbf{L}$-linearly, but not $\mathbf{K}$-linearly for $\mathbf{K} \ne \mathbf{L}$. In view of the Hahn-Banach theorem [8] there are $\mathbf{L}$-linear continuous functionals e_j such that $e_j(x_j) = 1$. Then $B_j := \{y : y \in X_{\mathbf{L}}, |e_j(y)|_{\mathbf{L}} \le 1\}$ are canonical closed subsets in $X_{\mathbf{L}}$. If α is countable, then $C_1 := C \cap (\bigcap_{j\in\alpha} B_j) \in 2^X_\delta$ and $E_1 := \bigcap_{j\in\alpha} \lambda_j C_1$ is the closed $\mathbf{L}$-linear subspace of $X_{\mathbf{L}}$. We get the inverse mapping system $\{X/E_1, \pi^{E'_1}_{E_1}, \Omega\}$, where $E'_1 > E_1$ if and only if $E'_1 \subset E_1 \in \Omega$ and $E'_1 \ne E_1$, Ω is the family of such E_1, $\pi^{E'_1}_{E_1} : X/E'_1 \to X/E_1$ are $\mathbf{L}$-linear. Therefore, $X_{\mathbf{L}} = pr - \lim\{X/E_1\}$ is locally convex. On the other hand, $\mathbf{K}$ is a the locally convex space over $\mathbf{L}$, consequently, X is locally convex.

It remains to consider the case, when there exists $C \in \mathsf{B}_o$ or $C \in 2^X_\delta$ such that for it the corresponding E has $card(\alpha) > \aleph_0$. In view of Corollary 5 [7] for each $\epsilon > 0$ there exists $U \in \mathsf{B}_o$ such that $0 \le \rho(x, C) - \rho(x, cl(C+U)) < \epsilon$

for each $x \in X$. Let $z \in cl(sp_{\mathbf{K}}E) \setminus E \neq \emptyset$, then $\rho(z, E) > 0$. Suppose $z = \sum_{j\in\alpha} \lambda_j x_j$ and $x_j \in U$ for each $\lambda_j \neq 0$, where α is an ordinal due to the Kuratowski-Zorn lemma. Certainly this sum expressing z has only countable subset $j \in \beta \subset \alpha$ of $\lambda_j \neq 0$. Due to Lemma 3.3 for each normed space Z its κ-norm has extension to the norm in Z, since $\{z\} \in 2^Z_\delta$ for each $z \in Z$. If for each $\epsilon > 0$ there exists such U, then from the equality $\rho(\lambda y, \lambda cl(E + U)) = |\lambda|\rho(y, cl(E + U))$ for each $0 \neq \lambda \in \mathbf{K}$ it follows that $0 \le \rho(z, E) - \rho(z, cl(E+U)) < \epsilon$, consequently, $\rho(z, E) < \epsilon$, since $\lim_j \lambda_j = 0$ and $cl(E + U)$ contains $\sum_{\beta\ni j\ge\alpha_0} \lambda_j x_j$ for some $\alpha_0 \subset \alpha$ of the cardinality $card(\alpha_0) < \aleph_0$ and each $\mathbf{K}^{\alpha_0}$ is the normed space. Since ϵ is arbitrary, then $z \in E$ and inevitably $cl(sp_{\mathbf{K}}E) = E$, contradicting our assumption.

Therefore, for each $\epsilon > 0$ there exists $U \in \mathsf{B}_o$ such that $x_j \notin U$ for each $j \in \alpha$ and $\bigcap_j \lambda_j U$ is the $\mathbf{K}$-linear subspace in X. Then such U would give $C \cap U$ with $\bigcap_j cl(\lambda_j(C \cap U)) =: E_0$ and $X = pr - \lim\{X/E_0\}$ again contradicting assumption on X, since E_0 are closed $\mathbf{K}$-linear subspaces in X and X/E_0 are normed spaces. Indeed, in view of continuity of addition in X for each $V \in \mathsf{B}_o$ there are $C \in \mathsf{B}_o$ and $U \in \mathsf{B}_o$ such that $cl(C + U) \subset V$. Therefore, X is the $\mathbf{K}$-locally convex space.

3.7. Note. The latter theorem shows, that there are topological vector spaces X, for which $(X, 2^X_\delta)$ are not κ-normable, for example, non locally convex metrizable X.

4 Embeddings of 2^X_o and 2^X_δ.

4.1. For a complete locally convex space X let S_X be either a family 2^X_o of all canonical closed subsets or a family 2^X_δ of all closed G_δ-subsets. As in §3.6 we consider the subfield $\mathbf{L}$ of the field $\mathbf{K}$ and the corresponding topological $\mathbf{L}$-linear space $X_{\mathbf{L}}$. Let P be a subset of the topological dual space $X^*_{\mathbf{L}}$ separating points of $X_{\mathbf{L}}$. Let μ be a non-negative Haar measure on $\mathbf{L}$. Two elements $A, B \in 2^X_\delta$ we call Ξ-equivalent if and only if $\mu(\pi(A \triangle B)) = 0$ for each $\pi \in \mathsf{P}$, where as usually $(A \triangle B) = (A \setminus B) \cup (B \setminus A)$. This equvalence is written as $A \Xi B$.

If X is over $\mathbf{R}$ or $\mathbf{C}$ we take μ such that $\mu([0, 1]) = 1$. In the non-Archimedean case there are two variants: the characteristic of the field $\mathbf{K}$ is $char(\mathbf{K}) = s > 0$; $char(\mathbf{K}) = 0$, then $\mathbf{K}$ is the extension of the field $\mathbf{Q_s}$ for the corresponding prime number s, then we take $s \neq p$. In view of the

Monna-Springer theorem 8.4 [9] there exists the non-trivial Haar measure $\mu : Bco(\mathbf{L}) \to \mathbf{Q_p}$ such that $\mu(B(\mathbf{L},0,1)) = 1$, where $Bco(\mathbf{L})$ is the algebra of clopen (closed and open) subsets in $\mathbf{L}$. By the definition of the Haar measure $\mu(z+A) = \mu(A)$ for each clopen compact subset A in $\mathbf{L}$ and each $z \in \mathbf{L}$ in the non-Archimedean case and for each Borel subset A and each $z \in \mathbf{L}$ in the classical case (that is, for X over fields $\mathbf{R}$ or $\mathbf{C}$). The measure μ has an extension from $Bco(\mathbf{L})$ on the σ-field $Bf(\mathbf{L})$ of Borel subsets. We say that A and $B \in 2^X_\delta$ are Υ-equivalent and write $A \Upsilon B$ if and only if $\int_{\pi(A \triangle B)} f(x)\mu(dx) = 0$ for each $\pi \in \mathsf{P}$ and μ-measurable f.

Theorem. *Let X be a κ-normed space. Then there exists a one-to-one continuous mapping Θ from S_X/Ξ into either $\mathbf{R}^{d(X)}$, when $\mathbf{K} = \mathbf{R}$ or $\mathbf{K} = \mathbf{C}$, or Ψ from S_X/Υ into ${\mathbf{Q_p}}^{d(X)}$ respectively, where $d(X)$ is the topological density of X, $\mathbf{p}$ is the prime number such that the field of $\mathbf{p}$-adic numbers $\mathbf{Q_p}$ is not contained in $\mathbf{K}$. Moreover, the restrictions $\Xi|_{2^X_o}$ and $\Upsilon|_{2^X_o}$ coincide with the equality of sets.*

Proof. For $\mathbf{K} = \mathbf{C}$ we take $\mathbf{L} = \mathbf{R}$ and $\mathbf{L} = \mathbf{R}$ for $\mathbf{K} = \mathbf{R}$. We consider the space $C^0(\mathbf{L}, Z)$ of continuous functions $f : \mathbf{L} \to Z$, where either $Z = \mathbf{R}$ or $Z = \mathbf{Q_p}$ respectively. We take a countable family U in $C^0(\mathbf{L}, Z)$ satisfying two conditions:

(i) the restriction $\mathsf{U}|_V$ is dense in $C^0(V, Z)$ for each canonical closed compact subset V in $\mathbf{L}$,

(ii) either $\|\mu^f\| < \infty$ in the case $\mathbf{L} = \mathbf{R}$ or $\sum_{j=1}^{\infty} \|\mu^f|_{B(\mathbf{L},x_j,1)}\| < \infty$ in the non-Archimedean case, where $\{B(\mathbf{K}, x_j, 1) : j \in \mathbf{N}\}$ is a disjoint clopen covering of $\mathbf{K}$, $x_j \in \mathbf{K}$, either $\|\mu^f\| := \int_{\mathbf{R}} |f(x)|\mu(dx)$ in the case $\mathbf{L} = \mathbf{R}$ or $\|\mu^f|_A\| := \sup_{B \subset A, B \in Bco(\mathbf{L})} |\mu^f(B)|$ for $A \in Bco(\mathbf{L})$, $\mu^f(A) = \int_A f(x)\mu(dx)$. To continue the proof we need the following lemma.

4.2. Lemma. *Let X be a κ-normable space. If $\pi \in X^*_{\mathbf{L}}$ and $E \in 2^X_\delta$, then $\pi(E) \in Bf(\mathbf{L})$, where $Bf(\mathbf{L})$ is the σ-field of Borel subsets in $\mathbf{L}$.*

Proof. In view of §2.4 [13] about quotient mappings if A is open in $X_{\mathbf{L}}$, then $\pi(A)$ is open in $\mathbf{L}$, since $\mathbf{L}$ is spherically complete and $\pi(A) = (A+Y)/Y$ due to the Hanh-Banach theorem [8], $Y := coker(\pi)$ for each $\pi \in X^*_{\mathbf{L}}$. In view of Lemma 2 [7] each $E \in 2^X_\delta$ is a countable intersection of open subsets U_j such that $U_j \supset U_{j+1}$ for each $j \in \mathbf{N}$. Then $\pi(E) = \bigcap_{j \in \mathbf{N}}((U+j+Y)/Y)$, since $\pi(E) \subset \bigcap_j \pi(U_j)$, $[z] \in (E+Y)/Y$ if and only if there exists $z \in E$ such that $(z+Y)/Y = [z]$ if and only if $z \in \bigcap_j U_j$ and $(z+Y)/Y = [z] \in (U_j+Y)/Y$ for each j. Therefore, $\pi(E)$ is the Borel subset in $\mathbf{L}$.

Continuation of the **proof** of Theorem 4.1. In view of Lemma 4.2 for

each $E \neq C \in 2^X_o$ there exists $\pi \in \mathsf{P}$ such that $\pi(E) \neq \pi(C)$ and $f \in \mathsf{U}$ for which $\mu^f(\pi(E)) \neq \mu^f(\pi(C))$. Then $\mu^f(\pi(E)) = 0$ for each $\pi \in \mathsf{P}$ and each $f \in \mathsf{U}$ is equivalent to $\mu(\pi(E)) = 0$ for each $\pi \in \mathsf{P}$ in the classical case, since μ^f is absolutely continuous relative to μ for each $f \in C^0(\mathbf{L}, Z)$. By the diagonal mapping theorem 2.3.20 [13] and Lemma 2 [7] we have a one-to-one continuous mapping

$$F := \Delta_{(\pi \in \mathsf{P}, f \in \mathsf{U})} \mu^f(\pi(*)) : W \to \prod_{(\pi \in \mathsf{P}, f \in \mathsf{U})} Y_{(\pi,f)},$$

where either all $Y_{(\pi,f)}$ are equal to $\mathbf{R}$ for $W = (2^X_\delta/\Xi)$ or all equal to $\mathbf{Q_p}$ for $W = (2^X_\delta/\Upsilon)$. The minimal cardinality of P which separates points in $X_{\mathbf{L}}$ is equal to the topological density $d(X_{\mathbf{L}})$ of the space $X_{\mathbf{L}}$. Evidently $d(X) = d(X_{\mathbf{L}})$, since X and $X_{\mathbf{L}}$ are homeomorphic as topological spaces (without linear structure). Since $card(\mathsf{U}) = \aleph_0$ and $d(X) \geq \aleph_0$, then $card(\mathsf{U})card(X) = card(X)$. From the proof it follows, that for $Z = \mathbf{L} = \mathbf{R}$ we can take instead of U satisfying Conditions 4.1.(i, ii) simply $\mathsf{U} = \{g\}$, where $g(x) = e^{-x^2}$ for each $x \in \mathbf{R}$.

4.3. Corollary. *There exists a one-to-one continuous mapping Θ of 2^X_o and $2^X_\delta/\Xi$ into $[0,1]^{d(X)}$ for $\mathbf{K} = \mathbf{R}$ and $\mathbf{K} = \mathbf{C}$ or Ψ of 2^X_o and $2^X_\delta/\Upsilon$ into ${\mathbf{Z_p}}^{d(X)}$ in the non-Archimedean case.*

Proof. Take in the proof of Theorem 4.1 in Condition (ii) $\|\mu^f\| \leq 1$ for the real or complex field and $\sum_j \|\mu^f|_{B(\mathbf{L},x_j,1)}| \leq 1$ in the non-Archimedean case.

4.4. Note. As shows a particular case of a normed space X such embeddings Θ and Ψ produce in general the weaker topologies inherited from the Tychonoff product topology in $W := \prod_{(\pi \in \mathsf{P}, f \in \mathsf{U})} \mathbf{L}$, than the initial one in the metric space $(2^X_o, D)$, since W is not metrizable for $d(X) > \aleph_0$. In general F is only continuous, but F^{-1} may be discontinuous.

The topology in 2^X_o or in $2^X_\delta/\Xi$ or in $2^X_\delta/\Upsilon$ inherited from W we call the h-weak topology and denote τ_w.

In the non-Archimedean case the κ-norm ρ induces the ultrametric D in 2^X_o or 2^X_δ, hence $(2^X_o, D)$ and $(2^X_\delta, D)$ are totally disconnected.

4.5. Corollary. *If X is over a non-Archimedean field, then 2^X_o and $2^X_\delta/\Upsilon$ are totally disconnected in the h-weak topology.*

4.6 Corollary. *$(2^X_o, \tau_w)$ and $(2^X_\delta/\Xi, \tau_w)$ and $(2^X_\delta/\Upsilon, \tau_w)$ have compactifications contained either in $[0,1]^{d(X)}$ for $\mathbf{K} = \mathbf{R}$ and $\mathbf{K} = \mathbf{C}$ or in ${\mathbf{Z_p}}^{d(X)}$ for the non-Archimedean field $\mathbf{K}$.*

4.7. Note. In accordance with Theorem 2.3.23 [13] there exists a homeomorphic embedding of (S_X, D) into $I^{w(S_X)}$, where $w(S_X)$ is the topological weight of S_X, where either $S_X = 2^X_o$ or $S_X = 2^X_\delta$, $I = [0,1]$. Therefore, $w(S_X) \ge d(X)$, since $w(I^{\mathsf{m}}) = \mathsf{m}$ for each $\mathsf{m} \ge \aleph_0$. Certainly equivalence relations Ξ and Υ are different, since in the classical case μ is atomless and in the non-Archimedean case μ on $Bf(\mathbf{L})$ is purely atomic (because of Ch. 7 [9]). Example 4.8 below and the results given above show, that functionals $\{\mu^f(\pi(*)) : f, \pi\}$ do not separate points $E \in 2^X_\delta$ and closed subsets $F \subset 2^X_\delta$ with $E \notin F$, that is, there are such E and F, which are not separated.

4.8. Example. $(2^{\mathbf{R^n}}_o, D)$ is not separable and it is dense in $(2^{\mathbf{R^n}}_\delta, D)$, since countable unions of closed parallelepipeds $\{\prod_{i=1}^n [a_i, b_i]\}$ with $a_i, b_i \in \mathbf{Q}$ are dense in $2^{\mathbf{R^n}}_o$. It can be used the Souslin number $hc(Y)$ of a topological space Y, which is the least cardinal such that every subset A of Y consisting exclusively of isolated points has $card(A) \le hc(Y)$, so $w(Y) \ge hc(Y)$ (see §1.7.12 [13]). On the other hand, $2^{\mathbf{R^n}}_\delta$ has the metric inherited from $\mathbf{R^n}$. Therefore,

$$\aleph_0^{\aleph_0} = \mathsf{c} = w(2^{\mathbf{R^n}}_\delta) = w(2^{\mathbf{R^n}}_o) > w(\mathbf{R^n}) = \aleph_0.$$

In the non-Archimedean case of the locally compact infinite field with the non-trivial valuation countable unions of balls $\{B(\mathbf{K^n}, x_j, r_j) : j \in \Omega, r_j \in \Gamma_{\mathbf{L}}\}$ are dense in $2^{\mathbf{L^n}}_\delta$, where Ω is a countable dense subset in $\mathbf{L^n}$ and $\Gamma_{\mathbf{L}} := \{|z|_{\mathbf{L}} : 0 \ne z \in \mathbf{L}\}$ is discrete in $(0, \infty)$. Therefore, $w(2^{\mathbf{L^n}}_\delta) = w(2^{\mathbf{L^n}}_o) = \mathsf{c}$, since there exists a family of the cardinality $\aleph_0$ of disjoint balls in $\mathbf{R^n}$ and in $\mathbf{L^n}$ with $\inf_{j \in \Omega} r_j > 0$, hence $c(\mathbf{L^n}) = \aleph_0$ and $hc(2^{\mathbf{L^n}}_o) = \mathsf{c}$.

4.9. Theorem. *There exists a homeomorphic embedding of 2^X_δ either into $I^{2^{w(X)}}$ for a κ-normable topological vector space X over $\mathbf{R}$ or $\mathbf{C}$, or into $\{0,1\}^{2^{w(X)}}$ for X over a non-Archimedean infinite field with a non-trivial valuation, where $\{0,1\}$ is a two-element dicrete set.*

Proof. Each open subset is a union of some subfamily of elements of a base, hence using closures of open subsets we get $card(2^X_o) \le w(X)^{w(X)}$. Therefore, $w(2^X_o) \le 2^{w(X)^{w(X)}} = 2^{2^{w(X)}}$, since $w(X) \ge \aleph_0$. In view of Lemma 2 [7] $w(2^X_\delta) \le w(2^X_o)^{\aleph_0}$. For the κ-normed space $(X, 2^X_\delta, \rho)$ the base of the topology of 2^X_o has the cardinality no greater, than $w(X)^{w(X)} = 2^{w(X)}$, since $w(X) \ge \aleph_0$ and $\rho(x, C)$ is uniformly continuous by $x \in X$ for each fixed $C \in 2^X_\delta$.

4.10. Remark. Then $w(X) \le w(2^X_\delta) \le 2^{w(X)}$, since $w(2^X_\delta) \ge w(X)$ and $card(2^X_o) \ge w(X)$.

5 Applications of κ-normed spaces.

5.1. Theorem. *Let X be a complete κ-normable locally convex space over the field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ and let be given a continuous mapping $f : \mathbf{R} \times (2^X_\delta \cup X) \to 2^X_\delta \cup X$ such that $f(t, 2^X_\delta) \subset 2^X_\delta$, $f(t, X) \subset X$ and $\rho_X(f(t,x), f(t,A)) \le C_A \rho_X(x, A)$ for each $A \in 2^X_\delta$ and $x \in X$, where a constant $C_A > 0$ may depend on A, $f(t, M) = M$ for each $t \in \mathbf{R}$ and $\mathbf{K}$-linear subspace M from the family F such that $\mathsf{F} \subset 2^X_\delta$, $M_1 \cap M_2 \in \mathsf{M}$ for each M_1 and $M_2 \in \mathsf{F}$, $\bigcap_{M \in \mathsf{F}} M = \{0\}$. Then the differential equation $dx(t)/dt = f(t, x(t))$ (or $dA(t)/dt = f(t, A(t))$) with the initial condition $x(0) = x_0 \in X$ (or $A(0) = A_0 \in 2^X_\delta$ for a convex balanced subset $A_0 - x_0$ for some $x_0 \in X$) has the unique solution $x : \mathbf{R} \to X$ (or $A : \mathbf{R} \to 2^X_\delta$ for $\mathsf{F} = 2^X_\delta$ correspondingly).*

Proof. (I). Let M be a $\mathbf{K}$-linear subspace such that $M \in \mathsf{F}$ (see the end of §2.3). Then $\rho_X(x, M) = \|x + M\|_{X/M}$ is the norm in the quotient space X/M, where $x + M = \theta_M(x)$, $\theta_M : X \to X/M$ is a quotient mapping, $x \in X$, $(x + M) \in X/M$. The function $f_M(t, x) = f(t, x) + M$ satisfies the following condition: $\|f_M(t, x)\|_{X/M} = \rho_X(f(t, x), f(t, M)) \le C_M \rho_X(x, M) = C_M \|x + M\|_{X/M}$, since $f(t, M) = M$. Moreover, $f_M : X/M \to X/M$ is continuous. The equation $dx_M(t)/dt = f_M(t, x_M(t))$ with the initial condition $x_M(0) = x_0 + M$ has the unique solution $x_M(t)$ in the Banach space X/M. Evidently that in the particular case, when $\mathsf{F} = 2^X_{\delta,l}$ is the family of all $\mathbf{K}$-linear subspaces belonging to 2^X_δ, the conditions imposed on F are satisfied, therefore the conditions of this theorem are correct. If M_1 and $M_2 \in \mathsf{F}$, then $x_{M_1}(t) \cap x_{M_2}(t) = x_{M_1 \cap M_2}(t)$, since $(f(t,x) + M_1) \cap (f(t,x) + M_2) = f(t,x) + (M_1 \cap M_2)$. We have $\bigcap_{M \in \mathsf{F}} x_M(t) = x(t) \in X$ and $\bigcap_{M \in \mathsf{F}} f_M(t, x) = f(t, x)$, consequently, $dx(t)/dt = f(t, x)$.

(II). Let $A \in 2^X_\delta$ be convex and balanced, then $M := (\bigcap_n \lambda_n A) \in 2^X_{\delta,l}$, where $0 \ne \lambda_n \in \mathbf{K}$ for each $n \in \mathbf{N}$ and $\lim_{n\to\infty} \lambda_n = 0$. Consequently, $\theta_M(A) = (A + M) \in 2^{X/M}_\delta$, in addition $A + M$ is bounded in the Banach space X/M, since X is complete. From the inequality $\rho_X(f(t,x), f(t,A)) \le C_A \rho_X(x, A)$ it follows that $D(f(t, A_1), f(t, A_2)) = \bar\rho_X(f(t, A_1), f(t, A_2)) + \bar\rho(f(t, A_2), f(t, A_1)) \le [C_{A_1} \bar\rho_X(A_2, A_1) + C_{A_2} \bar\rho_X(A_1, A_2)] \le D(A_1, A_2) C$, where $C = \max(C_{A_1}, C_{A_2})$. Then $\int_{t_1}^{t_2} [f(\tau, A) + M] d\tau = \int_{t_1}^{t_2} f(\tau, A) d\tau + M$, since $tM = M$, where $t_2 > t_1$, $t := t_2 - t_1$, $\int_{t_1}^{t_2} E(\tau) d\tau := \{\phi(t_1, t_2) := \int_{t_1}^{t_2} \psi(\tau) d\tau$, $\psi(\tau) \in E(\tau)\}$, $E(\tau) := f(\tau, E) = \{f(\tau, x) : x \in E\}$, $\psi(\tau) = f(\tau, x)$. In Lemma 8 [7] it was proved that the function $D(A, B) := \bar\rho(A, B) + \bar\rho(B, A)$

on $(2^X_\delta)^2$ is the metric satisfying additional conditions $D(A\hat{+}B, C\hat{+}E) \le D(A,C) + D(B,E)$ (or with the maximum instead of the sum in the non-Archimedean case) $D(\lambda A, \lambda C) = |\lambda| D(A,C)$ for each $0 \ne \lambda \in \mathbf{K}$, where $A\hat{+}B := cl(A+B)$. Using the approximation of continuous functions by simple (step) functions in the space $L^1([a,b],X)$ for $a \le t_1 < t_2 \le b$ we get:

$D(\int_{t_1}^{t_2} f_M(\tau, A_1)d\tau, \int_{t_1}^{t_2} f_M(\tau, A_2)d\tau) \le C(t)D(A_1, A_2)$,

that is, the mapping $F_M(t_1,t_2,A) := \int_{t_1}^{t_2} f_M(\tau, A)d\tau$ is contracting for $Ct < 1$. Then in each interval $[t_1,t_2] \ni 0$ with $t = t_2 - t_1 < C^{-1}$ there exists the unique solution $(A+M)(t)$ in X/M, where $(A+M)(t) = A(t) + M$. Since $M \subset A$, then $A+M = A$, that is, $A(t)$ is a solution with the initial condition $A_0 = A(0)$, where $M = \bigcap_n \lambda_n A_0$. Using finite coverings of segments $[a,b]$ and solving analogous tasks with initial conditions in intermediate points and sewing solutions we get the solution in each segment $[a,b] \ni 0$. Evidently, the second part of this theorem is accomplished for more general functions f with $f(t,M) = M$ for each $t \in \mathbf{R}$ and a given one subspace M such that $M \in 2^X_{\delta,l}$, where M is completely defined by the initial condition.

6 Approximations of representations of the interval orders with the help of κ-normed spaces.

6.1. Remarks. Mappings of interval orders play an important role in mathematical economics. In papers [2, 3, 4, 5] a problem of representations of interval orders was began to study naively with the help of pairs of semicontinuous and continuous functions. It was proved the existence of such pairs and examples were given, but this field remains investigated only a little. The problem of searching of all such pairs and investigations of cases, when it is possible to use one function instead of the pairs remains actual. For economic theories it is important to search for the best approximation.

If T is a preordered set or a set with an interval order, then its representation is searched by a function $f: T \to \mathbf{R}$, which preserve order, or a pair of functions $f, g: T \to \mathbf{R}$ such that $g(x) \ge 0$ for each $x \in T$ and $x_1 < x_2$ if and only if $v(x_1) + \sigma(x_1) < v(x_2)$. This section is devoted to the investigation of this problem with the help of κ-normed spaces.

Recall that by the definition the mapping $f : T \to \mathbf{R}$ represents T if

and only if $f : T \to \mathbf{R}$ is order-preserving. Certainly not for every T such f exists. Therefore, approximations are frequently more valuable, than the precise results, which may be nonexistent.

6.2. Definitions and notes. Let T be a preordered set and U_c be a family of linearly ordered countable subsets t in T and B_c be a subfamily of $t \in \mathsf{U}$ such that there are a_t and $b_t \in T$ with $a_t \le x$ and $x \le b_t$ for each $x \in t$. When the condition of countability of t is dropped we denote the corresponding families by U and B. The family of closed subspaces t in T we denote by C. There can be considered another families F of subsets in T.

Let $C_b(t, \mathbf{R})$ denotes a family of functions $f : t \to \mathbf{R}$ such that $\|f\| := \sup_{x \in t} |f(x)| < \infty$. If T is supplied with a topology τ and each $t \subset T$ is considered in a topology inherited from T, then $C^0_b(t, \mathbf{R})$ denotes a subspace of continous functions f in $C_b(t, \mathbf{R})$.

6.3. Theorem. *For each family* F *there are* κ*-normable spaces* $(X_{\mathsf{F}}, S_{X_{\mathsf{F}}})$ *both for the cases of a family of all canonical closed subsets* $S_{X_{\mathsf{F}}} = 2^{X_{\mathsf{F}}}_o$ *and a family of closed* G_δ*-subsets* $S_{X_{\mathsf{F}}} = 2^{X_{\mathsf{F}}}_\delta$.

Proof. Each space $C_b(t, \mathbf{R})$ and $C^0_b(t, \mathbf{R})$ is normed and hence κ-normed. The first space is the particular case of the second, when the set t is supplied with the discrete topology. Then $P_{\mathsf{F}} := \prod_{t \in \mathsf{F}} Y_t$ is the κ-normable space by a κ-norm ρ on $(X_{\mathsf{F}}, 2^{X_{\mathsf{F}}}_o)$, where P_{F} is in the product locally convex topology (see Theorem 14 [7]). If we take S_{F} as a subspace of P_{F} satisfying Conditions $(a - d)$ of Proposition 13 [7], then we get a κ-norm ρ on $(S_{\mathsf{F}}, 2^{S_{\mathsf{F}}}_\delta)$. The topology of the latter locally convex space S_{F} satisfies conditions of Theorem 3.2 above.

In view of Theorem 6 [7] the quotient space of the κ-normed space is κ-normed, then it can be taken into account an equivalence relation E in P_{F} and S_{F} generated by equalities

$$(i)\ Y_{t_1 \cap t_2} = Y_{t_1} \cap Y_{t_2}$$

in the case of $C_b(t, \mathbf{R})$. On the other hand, in the case of $C^0_b(t, \mathbf{R})$ the equality (i) is satisfied, when each $f \in C^0_b(t_1 \cap t_2, \mathbf{R})$ has a continuous bounded extension on $t_1 \cup t_2$. This is the case, for example, for F consisting of closed subspaces t in a normal topological space T (see Tietze-Uryson theorem 2.1.8 [13]). When Condition (i) is satisfied we can take either $X_{\mathsf{F}} = P_{\mathsf{F}}/\mathsf{E}$ or $X_{\mathsf{F}} = S_{\mathsf{F}}/\mathsf{E}$, which fits conditions of this theorem.

6.4. Theorem. *Let* (X, S_X, ρ) *be a* κ*-normed space and* D *be a metric*

in S_X induced by ρ. Then for each $C_v \in S_X$ and each $\epsilon > 0$ there exists $\emptyset \neq C_\sigma \in S_X$ such that $D(cl(C_v + C_\sigma), C_v) < \epsilon$.

Proof. In view of Corollary 5 [7] for each $C_v \in S_X$ and each $\epsilon > 0$ there exists an open neighbourhood U of zero in X such that $0 \le \rho(f, C_v) - \rho(f, cl(C_v + U)) < \epsilon$ for each $f \in X$. We take $C_\sigma \in S_X$ such that $\emptyset \neq C_\sigma \subset U$ and $0 \in C_\sigma$. Then $\rho(f, cl(C_v + C_\sigma)) \ge \rho(f, cl(C_v + U))$ for each $f \in X$ due to the monotonicity axiom of ρ. Therefore, $0 \le \rho(f, C_v) - \rho(f, cl(C_v + C_\sigma)) < \epsilon$ for each $f \in X$. If $f \in cl(C_v + C_\sigma)$, then $\rho(f, cl(C_v + C_\sigma)) = 0$ due to the inclusion axiom. On the other hand, $\rho(f, C_v) \ge \rho(f, cl(C_v + C_\sigma))$ due to the monotonicity axiom. Hence $\bar{\rho}(C_v, cl(C_v + C_\sigma)) = \sup_{f \in C_v} \rho(f, cl(C_v + C_\sigma)) = 0$. In view of the triangle inequality axiom $\rho(f, C_v) \le \rho(f, cl(C_v + C_\sigma)) + \bar{\rho}(cl(C_v + C_\sigma), C_v)$. Then $0 \le \rho(f, C_v) < \epsilon$ for each $f \in cl(C_v + C_\sigma)$, consequently, $D(C_v, cl(C_v + C_\sigma)) = \bar{\rho}(C_v, cl(C_v + C_\sigma)) + \bar{\rho}(cl(C_v + C_\sigma), C_v) = \sup_{f \in cl(C_v + C_\sigma)} \rho(f, C_v) \le \epsilon$.

6.5. Theorem. *Let T be a preordered set or T be a set with an interval order and F be a family of subsets of T such that there exists a κ-normed space $(X_{\mathsf{F}}, 2_\delta^{X_{\mathsf{F}}}, \rho)$. Suppose there is a function $f : T \to \mathbf{R}$ which represents each $t \in \Lambda$ and $f|_t$ is bounded, where $card(\Lambda) < \aleph_0$ and $\Lambda \subset \mathsf{F}$.*

(1). *Then there exists $C \in 2_\delta^{X_{\mathsf{F}}}$ such that $f \in C$.*

(2). *Moreover, C can be chosen such that $\pi_t(C) = f|_t$ for each $t \in \Lambda$.*

Proof. Let X_{F} be a κ-normed space from Theorem 6.3 with $S_{X_{\mathsf{F}}} = 2_\delta^{X_{\mathsf{F}}}$. When Λ is a finite family we can take $2_o^{X_{\mathsf{F}}}$ also instead of $2_\delta^{X_{\mathsf{F}}}$, but the Condition (2) in general may be unsatisfied. Therefore, we have the κ-normed space $(X_{\mathsf{F}}, 2_\delta^{X_{\mathsf{F}}}, \rho)$. In view of Theorem 6 [7] the κ-norm ρ on $(X_{\mathsf{F}}, 2_\delta^{X_{\mathsf{F}}})$ generates a norm on Y_t by the formula $\tilde{p}(\pi_t(f)) = \rho(f, Y_t)$, since $Y_t \in 2_\delta^{X_{\mathsf{F}}}$, where $\pi_t : X_{\mathsf{F}} \to Y_t$ is the quotient mapping. This norm is equivalent to the initial one. On the other hand, $\pi_t(f) = f|_t$ and $\{\pi_t(f)\}$ is a G_δ-subset in Y_t. Hence $C = \bigcap_{t \in \Lambda} \pi_t^{-1}(\{\pi_t(f)\})$ is in $S_{X_{\mathsf{F}}}$ and $f \in C$. This C also satisfies Condition (2) for $S_{X_{\mathsf{F}}} = 2_\delta^{X_{\mathsf{F}}}$.

In the case of $card(\Lambda) < \aleph_0$ and $S_{X_{\mathsf{F}}} = 2_o^{X_{\mathsf{F}}}$ we can take balls $B_t := \{y : y \in Y_t, p_t(y_t - \pi_t(f)) \le \epsilon_t\}$ with $\infty > \epsilon_t > 0$ for each $t \in \Lambda$, then $C = \bigcap_{t \in \Lambda} \pi_t^{-1}(B_t)$ satisfies Condition (1). For $card(\Lambda) = \aleph_0$ and $S_{X_{\mathsf{F}}} = 2_\delta^{X_{\mathsf{F}}}$ we can take $C = \bigcap_{t \in \Lambda} \pi_t^{-1}(B_t)$ satisfying Condition (1).

6.6. Corollary. *Let $\mathsf{F} \in \{\mathsf{U}, \mathsf{U_c}, \mathsf{B}, \mathsf{B_c}, \mathsf{C}\}$, then there exists a κ-normed space $(X_{\mathsf{F}}, 2_\delta^{X_{\mathsf{F}}}, \rho)$ such that for each representation function f of a preordered or an interval ordered set T on a countable subfamily Λ of a family $\mathsf{F} \in \{\mathsf{U}, \mathsf{U_c}, \mathsf{B}, \mathsf{B_c}, \mathsf{C}\}$, for which the restriction $f|_t$ is bounded and represents t for*

each $t \in \Lambda$, there exists $C \in 2_\delta^{X_{\mathsf{F}}}$ with $f \in C$ and $\pi_t(C) = f|_t$ for each $t \in \Lambda$.

6.7. Note. For the family $\mathsf{B_c}$ the following condtion: "$f|_t$ is bounded" is automatically satisfied, since $-\infty < f(a_t) \le f(x) \le f(b_t) < \infty$ for each $x \in t$, where $a_t = \inf_{a\in t} a$ and $b_t := \sup_{b \in t} b$. The family $\mathsf{U_c}$ corresponds to approximation of f on each countable family of linearly ordered countable subsets.

The family C corresponds to approximation of f on countable families of closed subsets. In particular there can be taken a subfamily of compact subsets. This construction can be refined (see below). In general in the segment $[a,b] \subset \mathbf{R}$ there can be embedded $\mathsf{c} = card(\mathbf{R})$ distinct subsets, which are linearly ordered. For example, the ring $\mathbf{Z_p}$ of integer p-adic numbers x, that is, $x = \sum_{j=0}^\infty x_j p^j$, where $x_j \in \{0,1,...,p-1\}$, p is the prime number. The ring $\mathbf{Z_p}$ is linearly ordered by the relation $x \triangle y$ if and only if $x_0 = y_0$, $x_1 = y_1$,..., $x_n = y_n$ and $x_{n+1} < y_{n+1}$. As it is well-known $\mathbf{Z_p}$ is totally disconnected and homeomorhic with the Cantor set $\{0,1\}^{\aleph_0}$, where $\{0,1\}$ is the discrete two-element set. The non-Archimedean metric in $\mathbf{Z_p}$ can correspond to economic models with the reciprocal dependence on a parameter $x \in T$, $x \mapsto x^{-1}$. There are c pairwise distinct subsets in $[a,b]$, which are homeomorhic with $\mathbf{Z_p}$. In $[a,b]$ there exist also c pairwise distinct subsets (either countable or) of the cardinality c with the linear order inherited from the field $\mathbf{R}$.

6.8. Theorem. *Sets C in Theorem 6.5 can be chosen such that*

(i) for each $g \in C$ and $t \in \Lambda$ a restriction $g|_t$ is a nondecreasing function.

Proof. Let C be a set from Theorem 6.5. We construct from it a new set denoted by V satisfying Condition (i). Since X_{F} is a complete locally convex space and there exists a topological dual space X_{F}^* of continuous linear functionals on X_{F} such that X_{F}^* separates points of X_{F} by the Hahn-Banach theorem [8]. In particular a mapping $G_{x_1,x_2}(f) := f(x_1) - f(x_2)$ for a fixed pair $\{x_1,x_2\} \subset T$ is a continuous linear functional on X_{F}. Then $C^+_{x_1,x_2} := \{g : g \in C, g(x_1) \le g(x_2)\}$ is a closed subset in C, since $g(x_1) \le g(x_2)$ is equivalent to $G_{x_1,x_2}(g) \le 0$. On the other hand, $C^\epsilon_{x_1,x_2} := \{g : g \in C, G_{x_1,x_2}(g) < \epsilon\}$ is open in C for each $\epsilon > 0$. Then $\bigcap_{n=1}^\infty C^{1/n}_{x_1,x_2} = C^+_{x_1,x_2}$, consequently, $C^+_{x_1,x_2} \in 2_\delta^{X_{\mathsf{F}}}$. We take $V = \bigcap_{(x_1<x_2; x_1,x_2\in t, t\in\Lambda)} C^+_{x_1,x_2}$. Since $card(\Lambda) \le \aleph_0$ and $card(\bigcup_{n=2}^\infty \Lambda^n) = \aleph_0$, then $V \in 2_\delta^{X_{\mathsf{F}}}$. It is only necessary to verify that $V \ne \emptyset$. It is evident, when f is given, since $f \in C^+_{x_1,x_2}$ for each $x_1 < x_2$ in t.

6.9. Theorem. *Let* Λ *be a countable subfamily of* F*, where* $\mathsf{F} \in \mathsf{U_c}$*. Suppose* $\emptyset \ne C \in 2^{X_\mathsf{F}}_\delta$ *and* $\pi_t(C)$ *has a nonempty interior for each* $t \in \Lambda$*, where* π_t *are quotient mappings from Theorem 6.3, then* $\bigcap_{x_1<x_2;x_1,x_2\in t;t\in\Lambda} C^+_{x_1,x_2} = V \ne \emptyset$ *and* $V \in 2^{X_\mathsf{F}}_\delta$.

Proof. In §6.8 it was proved that $V \in 2^{X_\mathsf{F}}_\delta$. If $x_1 < x_2$ in t, where $t \in \Lambda$, then $\|G_{x_1,x_2}\|_{Y^*_t} \le 2$, hence $C^+_{x_1,x_2}$ is contained in $\pi_t^{-1}(U^0)$, where $U^0_t = U^0$ is the absolute polar of the ball $U = B(Y^*_t, 0, 2)$ in Y^*_t, $U^0 \subset Y_t$, (Y_t, Y^*_t) is the dual pair. In view of the Alaoglu-Bourbaki theorem (9.3.3) [8] U^0 is $\sigma(Y_t, Y^*_t)$-compact. Since $\pi_t(C)$ contains the ball $B(Y_t, g_t, r_t)$ we can take $\tilde{C} \in 2^{X_\mathsf{F}}_\delta$ such that $\pi_t(\tilde{C}) = B(Y_t, g_t, r_t)$ for each $t \in \Lambda$. In view of the Hahn-Banach theorem in its geometric form (8.5.3) [8] the ball $B(Y_t, g_t, r_t)$ is weakly closed, hence $\pi_t(\tilde{C})$ is a compact subset of U^0. In view of Theorem 3.1 we can choose a family of seminorms p_t defining topology of X_F such that $\sup_{t\in\Lambda} p_t(x) =: p(x) < \infty$ is a continuous seminorm in X_F, since $card(\Lambda) \le \aleph_0$. So instead of $t \in \Lambda$ we can take $t_0 = \bigcup_{t\in\Lambda} t$. Then $\pi_{t_0}(\tilde{C})$ is a weakly compact subset of $U^0_{t_0}$, where $U^0_{t_0}$ is for the space Y_{t_0}. For each $x_1 < x_2 < ... < x_n$ there evidently exists $g : T \to \mathbf{R}$ such that $g \in X_\mathsf{F}$ and $g(x_1) < g(x_2) < ... < g(x_n)$. Then $\pi_{t_0}(\tilde{C}^+_{x_1,x_2})$ are weakly closed subsets of $\pi_{t_0}(\tilde{C})$. Since $(\bigcap_{x_1<x_2;x_1,x_2\in t;t\in\Lambda} \pi_{t_0}(\tilde{C}^+_{x_1,x_2})) \ne \emptyset$, then $\bigcap_{(x_1<x_2;x_1,x_2\in t;t\in\Lambda)} \tilde{C}^+_{x_1,x_2} \ne \emptyset$.

6.10. Note. The construction above can be generalized to encompass the case of unbounded functions. For this we substitute $\|f\| = \sup_{x\in t} |f(x)|$ on another seminorm $\|f\| = \sup_{x\in t} |f(x)|\phi(x)$, where $\phi : t \to (0, \infty)$ is a fixed function such that $\phi(x) > 0$ for each $x \in t$. We can take different ϕ and $t \in \mathsf{F}$ in particular such that $1/\phi$ is unbounded. With sufficiently large family $\{\phi\}$ we can take into account unbounded $f : T \to \mathbf{R}$.

When T is either a subset of $\mathbf{R}$ in a topology inherited from $\mathbf{R}$ or $T \subset \mathbf{Q_p}$ in a topology iherited from $\mathbf{Q_p}$, then we can consider also functions satisfying the Lipshitz condition

$$(i)\ |g(x_1) - g(x_2)| \le C_1|x_1 - x_2|,$$

where $C_1 = const > 0$ and also satifying condition

$$(ii)\ (g(x_2) - g(x_1)) \ge C_2(x_2 - x_1)$$

for each $x_2 > x_1$ in t, where $t \in \Lambda$ (in (ii) for $T \subset \mathbf{R}$). Then Theorem 6.9 can be modified that to satisfy Condition (ii), where C_2 can be dependent on t.

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